Warmup: Let $S=\{1,2,3,4\}$

Prove: $\forall x\in S, (\exists y\in S, \ |x-y|\geq 2.)$

Scratch work: If $x=1 \longrightarrow y=3$ or 4

$2 \longrightarrow y=4$

Case 1

$3 \longrightarrow y=1$

$4 \longrightarrow y=1$ or 2

Case 2

Pf: Consider any $x\in S$. Either $x\in\{1,2\}$, or $x\in\{3,4\}$.
We do each case separately.

Case 1: $x\in\{1,2\}$. Then let $y=x+2$.

Proved $\exists y\in S, |x-y|\geq 2$.

Then $y\in\{3,4\}$, so $y\in S$. And

$$|x-y| = |x-(x+2)| = 2.$$

So we've found a y that works.

Case 2: $x\in\{3,4\}$. Let $y=x-2$.

(—— Proof similar ——)

We have addressed both cases, so the statement
$\forall x\in S, \exists y\in S, |x-y|\leq 2$ is true. ∎

Proving existence statements

Proving $\exists x \in S,\ P(x)$	Proving $\forall x \in S,\ P(x)$
Provide an example, show it works.	Use abstract logic.

Ex: "There exists a positive integer which cannot be written as a sum of two squares of integers."

Pf: 3 cannot be written as a sum of two squares. ∎

Note: You don't have to find all the examples, just one!

Note:

Disproving $\exists x \in S,\ P(x)$	Disproving $\forall x \in S,\ P(x)$
Negation: $\forall x \in S,\ \sim P(x)$	Negation: $\exists x \in S,\ \sim P(x)$
Abstract Logic	Give a counterexample

Exercise: • Convert the following to quatifiers.

"There is no smallest positive real number."

Notation: Let $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ be the set of positive real numbers.

$$\text{"}\exists x \in \mathbb{R}_{>0}, (\forall y \in \mathbb{R}_{>0}, (x \leq y))\text{"}$$

There is a smallest positive real

$$\text{"}\sim \exists x \in \mathbb{R}_{>0}, (\forall y \in \mathbb{R}_{>0}, (x \leq y))\text{"}$$

← There is no smallest positive real number

$$\equiv \forall x \in \mathbb{R}_{>0}, (\exists y \in \mathbb{R}_{>0}, (x > y))$$

Common error:

$\sim \exists x \in \mathbb{R}_{>0}, \ldots$

$\equiv \forall x \in \mathbb{R}_{\leq 0}, \ldots$

Wrong!

Proof: See notes from last class!

Quantifiers in Calculus

Suppose $x_1, x_2, x_3, \ldots$ are real numbers.

We say the sequence converges to a limit $L \in \mathbb{R}$

if "no matter how close I want to get to L,
you can eventually get that close."

$\equiv$ "$\forall \varepsilon \in \mathbb{R}_{>0}$, the sequence eventually gets and stays within distance ε of L."

$\equiv$ "$\forall \varepsilon \in \mathbb{R}_{>0}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N},$
if $n > N$, then $|x_n - L| < \varepsilon$."

ε = how close I want to get to L,

N = the cutoff point in the sequence such that every term after that is within ε of L.